Consider a system consisting of (2) identical particles, 1 & 2.

$$\Psi = \Psi(x_1, x_2).$$

The operator $\hat{P}_{12}$ is the exchange / permutation operator.
permutes labels $1 \longleftrightarrow 2$

$$\hat{P}_{12} f(x_1, x_2) = f(x_2, x_1)$$

Fundamental fact: $\Psi(x_1, x_2)$ and $\Psi(x_2, x_1)$ are observationally equivalent

$$\Rightarrow \quad \hat{P}_{12}\Psi(x_1, x_2) = \Psi(x_2, x_1)$$

$$= \underset{\text{phase}}{e^{i\alpha}} \Psi(x_1, x_2)$$

$$\left\{ |\Psi(1,2)|^2 = |\Psi(2,1)|^2 \right.$$

However, if we apply $\hat{P}_{12}$ twice, we obtain the original func^n.

$$\hat{P}_{12}^{\,2} \Psi(x_1, x_2) = (e^{i\alpha})^2 \Psi(x_1, x_2)$$

$$= \Psi(x_1, x_2)$$

$$\Rightarrow \quad \boxed{e^{i\alpha} = \pm 1}$$

$\therefore$ for identical QM particles, we must have

$$\boxed{\hat{P}_{12}\Psi(x_1, x_2) = \underset{\text{antisymmetric}}{\overset{\text{symmetric}}{\pm}} \Psi(x_1, x_2)}$$

Exchange of particle labels $\Longrightarrow$ (± 1)

Note: this result ensures invariance of expectation values under particle relabelling/exchange

we have

$$\langle \hat{\Omega} \rangle_{\Psi} = \int dx_1 \int dx_2 \; \Psi(x_1, x_2)^* \hat{\Omega} \; \Psi(x_1, x_2)$$

Apply $\hat{P}_{12}$:

$$\hat{P}_{12} \langle \hat{\Omega} \rangle_{\Psi} = \int dx_1 \int dx_2 (\hat{P}_{12} \Psi^*) [\hat{P}_{12} \hat{\Omega}] (\hat{P}_{12} \Psi)$$

As $\hat{\Omega}$ must have form

$$\hat{\Omega} = \hat{f}(x_1) + \hat{f}(x_2) \qquad \text{eg} \quad (x_1 + x_2)$$

or

$$\hat{\Omega} = \hat{g}(|x_1 - x_2|) \qquad \text{eg} \quad (x_1 - x_2)^2$$

$$\Rightarrow \quad \boxed{\hat{P}_{12} \hat{\Omega} = \hat{\Omega}} \quad \Leftarrow \text{ Condition on physical operators (symmetric)}$$

$$\Rightarrow \quad \hat{P}_{12} \langle \Omega \rangle_{\Psi}$$

$$= \int dx_1 \int dx_2 \, [(\pm) \Psi^*] \hat{\Omega} [(\pm) \Psi]$$

$$= \int dx_1 \int dx_2 \quad \Psi^* \hat{\Omega} \, \Psi$$

$$= \langle \Omega \rangle_{\Psi} \quad \checkmark\checkmark$$

ie, $\boxed{\text{The expectation value is invariant wrt exchange of identical particles.}}$

Fundamental fact ..

2 classes of particle in universe

- Fermions ½ - integer spin ½ , 3/2 , ..

  n - particle wf is anti - symmetric wrt exchange of any 2 identical fermions.

  Ex e^- , p^+ , n^0.

- Bosons Integer spin 0, 1, ..

  n - particle wf is symmetric wrt exchange of any 2 identical bosons

  Ex Photons, ^2D nucleus, Higgs boson $(S=0)$ ...
  $(j=1)$ $(j=1)$

[Spin - statistics connection ?
No elementary proof, so we don't really understand ! (Feynman)]

Consider : 2 particles, 2 states ϕ_a, ϕ_b.

3 cases :

1) Distinguishable particles : { H - atom / D - atom
p^+ / e^-

$$\underbrace{\Psi = \phi_a(x_1) \cdot \phi_b(x_2)}_{\text{Product wf}}$$

ii) Identical bosons

$$\Psi = \frac{1}{\sqrt{2}} \left[\phi_a(x_1)\, \phi_b(x_2) + \phi_a(x_2)\, \phi_b(x_1) \right]$$

Check:

Normalization

$$\int dx_1 \int dx_2 \; \Psi^* \Psi$$

$$= \frac{1}{2} \int dx_1 dx_2 \left[\phi_a(1)^* \phi_b(2)^* + \phi_a(2)^* \phi_b(1)^* \right] \cdot \left[\phi_a(1)\, \phi_b(2) + \phi_a(2)\, \phi_b(1) \right]$$

$$= \frac{1}{2} \Big\{ \int dx_1 \; \phi_a^*(1)\, \phi_a(1) \cdot \int dx_2\, \phi_b(2)^* \phi_b(2)$$
$$+ \int dx_2 \; \phi_a(2)^* \phi_a(2) \cdot \int dx_1 \; \phi_b(1)^* \phi_b(1)$$
$$+ \int dx_1 \, \phi_a(1)^* \phi_b(1) \int dx_2 \, \phi_b(2)^* \phi_a(2)$$
$$+ \int dx_1 \, \phi_b(1)^* \phi_a(1) \int dx_2 \; \phi_a(2)^* \phi_b(2) \Big\}$$

$$= \langle \phi_a | \phi_a \rangle \langle \phi_b | \phi_b \rangle + | \langle \phi_a | \phi_b \rangle |^2$$

$$= 1 \qquad \begin{cases} \langle \phi_a | \phi_a \rangle = \langle \phi_b | \phi_b \rangle = 1 \\ \langle \phi_a | \phi_b \rangle = \underline{\underline{0}} \end{cases}$$

$$\hat{P}_{12} \cdot \Psi = \frac{1}{\sqrt{2}} \left[\phi_a(x_2)\, \phi_b(x_1) + \phi_a(x_1)\, \phi_b(x_2) \right]$$

$$= (+)\; \Psi \quad \checkmark$$

iii) Identical fermions

$$\Psi = \frac{1}{\sqrt{2}} \left[\phi_a(x_1)\, \phi_b(x_2) - \phi_a(x_2)\, \phi_b(x_1) \right]$$

We have

$$\hat{P}_{12}\, \Psi = \frac{1}{\sqrt{2}} \left[\phi_a(x_2)\, \phi_b(x_1) - \phi_a(x_1)\, \phi_b(x_2) \right]$$

$$= (-)\, \Psi \quad \checkmark$$

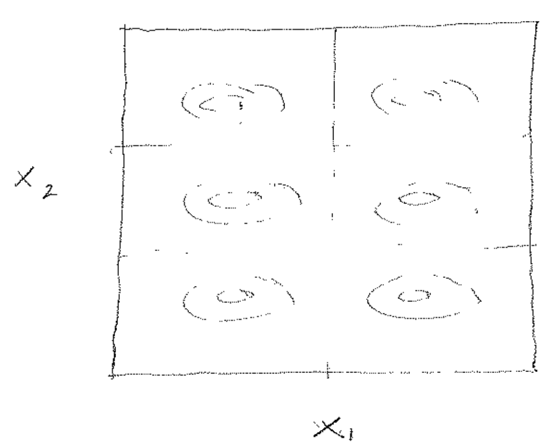
V. important point: symmetry / antisymmetry of identical particles introduces correlations, even in absence of interaction (potential) between particles.

Example: 2 PIB states: $\phi_a \equiv \phi_{n=2}$, $\phi_b = \phi_{n=3}$

(i) Distinguishable particles

$$\Psi(x_1, x_2) = \phi_2(x_1)\, \phi_3(x_2)$$

$$\Rightarrow \quad P(x_1, x_2) = |\Psi(x_1, x_2)|^2 = |\phi_2(x_1)|^2\, |\phi_3(x_2)|^2$$



As Ψ is a product, $P(x_1, x_2)$ factorizes

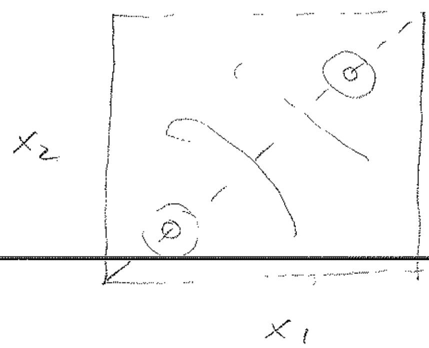
$\Rightarrow$ no correlations between particles

Ex $\langle x_1 x_2 \rangle = \langle x_1 \rangle \cdot \langle x_2 \rangle$

(ii) Bosons $\quad \Psi = \frac{1}{\sqrt{2}} [\phi_2(x_1)\phi_3(x_2) + \phi_2(x_2)\phi_3(x_1)]$

$P(x_1, x_2) = |\Psi|^2 \qquad$ no longer factorizes / separable.

Mathematica plot:



$P(x_1, x_2)$ is enhanced along diagonal, ie

$x_1 \sim x_2 \longleftarrow$ Particles in same part of box

ie, bosons tend to be closer together, $\simeq$ effective attraction. [but: NO interaction!]

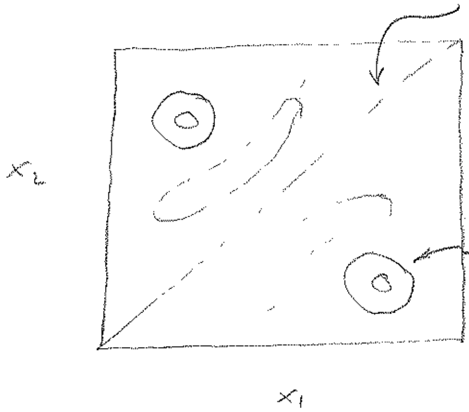
(iii) Fermions $\quad \Psi = \frac{1}{\sqrt{2}} [\phi_2(x_1)\phi_3(x_2) - \phi_2(x_2)\phi_3(x_1)]$

$P(x_1, x_2) = |\Psi|^2$.

NB If $x_1 = x_2$, $\boxed{\Psi = 0} \Rightarrow P(x_1, x_2 = x_1) = \underline{\underline{0}}$!

Node along diagonal: $\boxed{x_1 = x_2}$

Node $\quad (P = 0)$



Most prob: particles ~ opposite ends of box.

ie, fermions tend to be further apart than distinguishable particles

$\Rightarrow$ effective repulsion.

Mathematica: Calculate < Abs [$x_1 - x_2$] >

$$\langle |x_1 - x_2| \rangle_{Boson} < \langle |x_1 - x_2| \rangle_{Dist} < \langle |x_1 - x_2| \rangle_{Fermion}$$

Suppose $\phi_a = \phi_b$. [same state]

Fermions:

$$\Psi = \frac{1}{\sqrt{2}} \left[\phi_a(x_1)\, \phi_a(x_2) - \phi_a(x_2)\, \phi_a(x_1) \right]$$

$$= \underline{\underline{0}} \quad \text{identically}$$

$\Rightarrow$ 2 fermions cannot occupy same state

$\equiv$ Pauli exclusion principle

But: we know, 2 e^- can occupy (1s) orbital in He, say.

|| Must consider spin degrees of freedom as well.